Journal of Approximation Theory **114**, 152–163 (2002)
doi:10.1006/jath.2001.3631, available online at http://www.idealibrary.com on IDEAL®

On Polynomial Approximation of Square Integrable Functions on a Subarc of the Unit Circle

Li-Chien Shen

Department of Mathematics, University of Florida, Gainesville, Florida 32611-2082, U.S.A.

Communicated by Manfred v. Golitschek

Received May 23, 2000; accepted in revised form September 12, 2001;
published online December 21, 2001

Let E be an arc on the unit circle and let $L^2(E)$ be the space of all square integrable functions on E. Using the Banach–Steinhaus Theorem and the weak* compactness of the unit ball in the Hardy space, we study the L^2 approximation of functions in $L^2(E)$ by polynomials. In particular, we will investigate the size of the L^2 norms of the approximating polynomials in the complementary arc $\tilde{E}$ of E. The key theme of this work is to highlight the fact that the benefit of achieving good approximation for a function over the arc E by polynomials is more than offset by the large norms of such approximating polynomials on the complementary arc $\tilde{E}$.
© 2001 Elsevier Science (USA)
Key Words: Banach–Steinhaus Theorem; weak* compactness; reproducing kernel.

1. INTRODUCTION

Let E be an arc on the unit circle T: $|z|=1$. We define $L^2(E)$ to be the set of all measurable functions which are square integrable on the set E. The inner product over the set E is defined as

$$(f, g)_E = \frac{1}{2\pi}\int_E f\bar{g}\,|dz|$$

and the L^2 norm of f over the set E is $\|f\|_E = \sqrt{(f, f)_E}$. When $E=T$, we let $(f, g)=(f, g)_T$ and $\|f\|=\|f\|_T$. Let D be the unit disc, $|z|<1$, $H^2(D) = \{f\colon f$ is analytic in D and $f \in L^2(T)\}$, and $H^2(E)$ is the Hilbert space consisting of all the functions in $H^2(D)$ with the inner product $(f, g)_E$.

We now describe the main results of this work.

We first mention the following result concerning the polynomial approximation of functions in $L^2(E)$.

THEOREM 1.1. *Let $m(E)$ denote the length of E. If $m(E)<2\pi$ then, the set of polynomials are dense in $L^2(E)$.*

0021-9045/01 $35.00
© 2001 Elsevier Science (USA)
All rights reserved.

Variants of this theorem are well known; see, for instance Theorem 3.15 of [2, p. 90].

The following two results provide a sharp contrast to Theorem 1.1.

Let V_n be the set of polynomials of degree $\leqslant n$. For a given f in $H^2(D)$, let $\bar{p}_n(f)$ be the polynomial in V_n which provides the best L^2 approximation for f over the arc E:

$$\|\bar{p}_n(f)-f\|_E = \inf_{p \in V_n} \|p-f\|_E.$$

THEOREM 1.2. *Let*

$$B = \{f : f \in H^2(D) \text{ and } \sup_n \|\bar{p}_n(f)\|_E < \infty\}$$

Then B is of first category in $H^2(D)$.

Hence, for almost all the functions in $H^2(D)$, the process of the best approximation over the arc E will result in a large norm in the complementary arc $\tilde{E}$.

Let f be a function defined on T and let $f_{|E}$ denote the restriction of f on the arc E.

THEOREM 1.3. *Suppose $g \in L^2(E)$ and $g \neq f_{|E}$ for any $f \in H^2(D)$. If $\{p_n\}$ is a sequence of polynomials such that*

$$\|p_n - g\|_E \to 0, \tag{1.1}$$

Then

$$\sup_n \|p_n\| = \infty$$

We will study the reproducing kernel of $H^2(D)$ restricted to the arc E and obtain, as eigenfunctions, a class of polynomials which are orthogonal on arcs $E, \tilde{E}$ and the entire circle T. Theorem 1.2 follows from the Banach–Steinhaus Theorem and the properties of the eigenvalues of these polynomials.

Theorem 1.3 is a direct consequence of the weak* compactness of the bounded sets in $H^2(D)$-a special case of the Alaoglu Theorem [3, p. 66].

2. PROPERTIES OF THE EIGENVALUES AND EIGENFUNCTIONS OF THE REPRODUCING KERNEL RESTRICTED TO THE ARC E

In this section, we will study the basic properties of the orthogonal polynomials particularly suited for the study of the best L^2 approximation process over the arc E. Unlike the ones derived from the standard orthogonalization process (see, for instance, [5, Chapt. XI]), these are the eigenfunctions of the reproducing kernel of the unit disc restricted to the arc E. Most interestingly, they are also orthogonal over the complementary arc $\tilde{E}$ and this allows us to carry out the generalized Fourier series expansion of a function over the arcs E and $\tilde{E}$ using the same orthogonal sequence.

It is well known that the monomials z^n, $n = 0, 1, 2, \ldots$, form a complete orthonormal basis for the space $H^2(D)$. Let $f \in L^2(T)$ and

$$\hat{f}(n) = (f, z^n).$$

Then the projection of Pf of f onto $H^2(D)$ can be computed by the following formula,

$$Pf(z) = \sum_{n=0}^{\infty} \hat{f}(n)\, z^n = (1/2\pi) \sum_{n=0}^{\infty} \int_T f(w)\, \bar{w}^n\, |dw|\, z^n \tag{2.1}$$
$$= \frac{1}{2\pi} \int_T \overline{K(w, z)} f(w)\, |dw|,$$

where $z \in D$ and $K(w, z) = 1/(1 - w\bar{z})$.

Let E be an arc on T. We define, for $f \in L^2(T)$,

$$P(E) f(z) = \sum_{n=0}^{\infty} \hat{f}_E(n)\, z^n = \frac{1}{2\pi} \int_E \overline{K(w, z)} f(w)\, |dw|, \tag{2.2}$$

where $\hat{f}_E(n) = (f, z^n)_E$. For $f \in L^2(E)$, we extend f to T by defining the values of f to be identically zero in the complement of E. Then $P(E)$ maps $L^2(E)$ into $H^2(D)$.

In the following, we shall devote special attention to the properties of the following operator,

$$P_N(E)\, f(z) = \sum_{n=0}^{N} \hat{f}_E(n)\, z^n = 1/2\pi \int_E \overline{K_N(w, z)}\, f(w)\, |dw| \tag{2.3}$$

which maps f into the space V_n of the polynomials of degree less than or equal to N, where $K_N(w, z) = (1 - w^N \bar{z}^N)/(1 - w\bar{z})$.

We first establish

THEOREM 2.1. *Let N be a fixed positive integer and E be an arc with $m(E) < 2\pi$.*

(a) *The operator* $P_N(E)$ *is a non-negative Hermitian operator from* $L^2(E)$ *to* V_N.

(b) $P_N(E)$ *has exactly* $N+1$ *eigenvalues* $\{\lambda_{N,n}\}$, *and the eigenfunctions* $\{f_{N,n}\}$ *can be chosen to form orthonormal polynomials of degree* $\leqslant N$ *(In Corollary* 2.1 *we will show that* $\{f_{N,n}\}$ *are orthogonal simultaneously with respect to the inner products on* E *and* $\tilde{E}$*; i.e. with respect to different inner products).*

(c) *If* $P_N(E)\, f = \lambda f$, *then* $P_N(\tilde{E})\, f = (1-\lambda)\, f$.

(d) $0 < \lambda_{N,n} < 1$, $n = 0, 1, \ldots, N$.

(e) *Let* $\lambda_{N,N}$ *and* $\lambda_{N,0}$ *be, respectively, the largest and smallest eigenvalues of* $P_N(E)$. *Then*

$$\lim_{N \to \infty} \lambda_{N,N} = 1 \qquad \textit{and} \qquad \lim_{N \to \infty} \lambda_{N,0} = 0.$$

Proof. (a) Let $f \in L^2(E)$ and define $f = 0$ for $z \in \tilde{E}$. Then the Fourier coefficients of f are given by

$$\hat{f}(n) = 1/2\pi \int_0^{2\pi} f(e^{it})\, e^{-int}\, dt = \hat{f}_E(n).$$

From Parseval's identity, we conclude that

$$\|P_N(E)\, f\|_E^2 \leqslant \|P_N(E)\, f\|_T^2 = \sum_{n=0}^{N} |\hat{f}(n)|^2 \leqslant \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \|f\|_E^2. \tag{2.4}$$

This shows that $P_N(E)$: $L^2(E) \to V_N$ is bounded and of norm $\leqslant 1$. It is Hermitian because its kernel satisfies $K_N(w, z) = \overline{K_N(z, w)}$. And since

$$(P_N(E)\, f, f)_E = \sum_{n=0}^{N} |\hat{f}(n)|^2 \geqslant 0,$$

it is non-negative.

(b) Since the range of $P_N(E)$ is in V_N, the eigenfunctions of $P_N(E)$ belong to V_N. The conclusion follows immediately from the fact that $P_N(E)$: $V_N \to V_N$ is Hermitian.

(c) We note that for any polynomial f of degree $\leqslant N$,

$$P_N(T)\, f = f. \tag{2.5}$$

Therefore $K_N(w, z)$ is the reproducing kernel for V_N. Hence, if f is an eigenfunction of $P_N(E)$, f belongs to V_N and from (2.5),

$$
\begin{aligned}
(2.6)\quad (1-\lambda) f &= P_N(T) f - P_N(E) f \\
&= \frac{1}{2\pi}\int_T \overline{K_N(w, z)}\, f(w)\, |dw| - \frac{1}{2\pi}\int_E \overline{K_N(w, z)}\, f(w)\, |dw| \\
&= \frac{1}{2\pi}\int_{\tilde{E}} \overline{K_N(w, z)}\, f(w)\, |dw| \\
&= P_N(\tilde{E})\, f.
\end{aligned}
$$

(d) Since $P_N(E) f$ is in V_N, $\|P_N(E) f\|_E < \|P_N(E) f\|$; hence, we have strict inequalities in (2.4), so that it is clear that any eigenvalue of $P_N(E)$ is less than one. From (c), we see that if $\lambda = 0$, then $\lambda = 1$ is an eigenvalue for $P_N(\tilde{E})$ which again contradicts (2.4).

We recall that

$$\lambda_{N,N} = \sup_{f \in L^2(E)} \|P_N(E) f\|_E / \|f\|_E .$$

We will prove (e) by showing

$$\lim_{N\to\infty} \sup_f \|P_N(E) f\|_E / \|f\|_E = 1.$$

Without loss of generality, we assume

$$E = \{z \in T : |arg\ z| \leqslant a\}$$

Let

$$k_N(e^{it}) = (1/(N+1)) \left(\frac{\sin((N+1)\, t/2)}{\sin(t/2)}\right)^2$$

be the Nth Fejér kernel [2, p. 12]. Then $k_N(e^{it})$ is a trigonometric polynomial of degree N, $\|k_N\| = 1$ and

$$\lim_{n\to\infty} k_N(e^{it}) = 0 \tag{2.7}$$

uniformly on $\tilde{E}$. We add that $k_N(z)$ has a pole of order N at $z = 0$; it is not a polynomial. To remove the pole, we consider

$$h(e^{it}) = e^{iNt} k_N(e^{it}).$$

Then $h(z)$ is a polynomial of degree $2N$ and, from (2.5),

$$P_m(T)\,h = h \tag{2.8}$$

for $m \geqslant 2N$.

Let

$$h_E(e^{it}) = \begin{cases} 0, & \text{if } e^{it} \in \tilde{E} \\ h, & \text{if } e^{it} \in E \end{cases}$$

and $u = h - h_E$. The reader should keep in mind the dependence of u, h, h_E on N. Then, from (2.7),

$$\|u\| \to 0 \quad \text{and} \quad \|h_E\|_E \to 1 \quad \text{as} \quad N \to \infty. \tag{2.9}$$

Then, for $m \geqslant 2N$,

$$\begin{aligned} \|h_E - P_m(E)\,h_E\|_E &\leqslant \|h_E - P_m(E)\,h_E\| \\ &= \|-u + h - P_m(E)\,h\| \\ &= \|-u + P_m(T)\,h - P_m(E)\,h\| && \text{(from (2.8))} \\ &= \|-u + P_m(\tilde{E})\,h\| && \text{(from (2.6))} \\ &= \|-u + P_m(\tilde{E})\,u\| \\ &\leqslant 2\,\|u\|, \end{aligned}$$

and this implies that

$$|\,\|P_m(E)\,h_E\|_E - \|h_E\|_E| \leqslant 2\,\|u\|.$$

Hence, for all $m \geqslant 2N$ we deduce from (2.9),

$$|\,\|P_m(E)\,h_E\|_E \,/\, \|h_E\|_E - 1| \leqslant 2\,\|u\| \,/\, \|h_E\| \to 0$$

as $N \to \infty$. And this along with (c) implies that the sequences of the largest and the smallest eigenvalues tend to 1 and 0, respectively.

It is natural to consider analogous result for the case $N = \infty$, however, the method involved is completely different from the one above, we will treat it separately in Section 6.

The most remarkable property of the eigenfunctions of $P_N(E)$ is their unusually strong degree of orthogonality.

Corollary 2.1. *Let $f_{N,n}$, $n = 0, 1, 2, \ldots, N$, be the eigenfunctions of $P_N(E)$. Then,*

(a) *they are simultaneously orthogonal on the arcs* E, $\tilde{E}$, *and* T,

(b) *if* $\|f_{N,n}\|_E = 1$, *then*

$$\|f_{N,n}\| = \frac{1}{\sqrt{\lambda_{N,n}}} \qquad \textit{and} \qquad \|f_{N,n}\|_{\tilde{E}} = \sqrt{\frac{1}{\lambda_{N,n}} - 1}\,.$$

Proof. (a) The orthogonality assertion follows from the fact (Theorem 2.1(c)) that $f_{N,n}$ are the eigenfunctions of the Hermitian operators $P_N(E)$ and $P_N(\tilde{E})$.

(b) Since $K_N(w, z)$ is the reproducing kernel for the polynomials of degrees $\leqslant N$ and $\{f_{N,n}\}$ is a complete orthogonal basis for the space of the polynomials of degrees $\leqslant N$, we deduce that

$$K_N(w, z) = \sum_{n=0}^{N} \frac{f_{N,n}(w) \cdot \overline{f_{N,n}(z)}}{\|f_{N,n}\| \cdot \|f_{N,n}\|}.$$

On the other hand,

$$K_N(w, z) = \sum_{n=0}^{N} \lambda_{N,n} f_{N,n}(w) \overline{f_{N,n}(z)}.$$

For a fixed z, $K_N(w, z)$ is a polynomial of degree N and since the representation of any polynomial of degree $\leqslant N$ by $f_{N,n}$ is unique, the first assertion follows immediately by comparing these two expressions for $K_N(w, z)$. The norm of f on the arc $\tilde{E}$ can be computed from the equation

$$(f_{N,n}, f_{N,n}) = (f_{N,n}, f_{N,n})_E + (f_{N,n}, f_{N,n})_{\tilde{E}}$$

3. PROOF OF THEOREM 1.1

First we recall a well known theorem of F. Carlson [6, p. 185]: Let $F(z)$ be analytic in the half plane $Re\, z > -c$ for some $c > 0$. If

$$|F(z)| = O(exp(a\,|z|)),$$

where $0 < a < \pi$, and $F(n) = 0$ for every integer $n \geqslant 0$, then $F(z) \equiv 0$.

Without loss of generality, assume $E = \{z \in T : |arg\, z| \leqslant a\}$.

Suppose there exists an $f \in L^2(E)$ such that $(f, p)_E = 0$ for all polynomials p. Then $(f, z^n)_E = 0$ for all integers $n > 0$. Consider

$$F(z) = \frac{1}{2\pi} \int_{-a}^{a} f(e^{it})\, e^{-izt}\, dt.$$

Then f is an entire function of exponential type t, $t \leqslant a$ and $F(n)=0$ for all integers $n \geqslant 0$. This implies, according to the above result, $F(z)=0$ for all z and, in particular, $F(n)=0$ for all negative integers as well. Thus all the Fourier coefficients of f are zero and from the uniqueness theorem of the Fourier series, we deduce that $f=0$ *a.e.* on E. This established the result.

An alternative proof is provided by the referee and we sketch it as follows:

By Mergelyan's theorem [4, Chap. 20], the polynomials are dense in the continuous functions on E. Moreover, the continuous functions are dense in $L^2(E)$. Thus, the statement of Theorem 1.1 follows.

4. PROOF OF THEOREM 1.2

We define, for a given $f \in H^2(E)$,

$$\bar{p}_N = \sum_{n=0}^{N} (f, f_{N,n})_E \, f_{N,n}.$$

Then it is the least mean square approximation of f by polynomials of degrees $\leqslant N$ on the arc E.

Let N be a fixed positive integer. For every f in $H^2(E)$, we define $T_N f = \bar{p}_N$. Then T_N is a bounded linear operator from $(H^2(E), \|\ \|_E)$ to $(H^2(\tilde{E}), \|\ \|_{\tilde{E}})$. Moreover, the norm is $\|T_N\| = \sqrt{\frac{1}{\lambda_{N,0}} - 1}$ (This is an analogue of the Lebesgue constant [2, p. 47] for the approximation of the Fourier series by the partial sums). The proof follows immediately from the simple inequality:

$$\begin{aligned}
\|T_N f\|_{\tilde{E}}^2 &= \sum_{n=0}^{N} \left(\frac{1}{\lambda_{N,n}} - 1 \right) |(f, f_{N,n})_E|^2 \\
&\leqslant \left(\frac{1}{\lambda_{N,0}} - 1 \right) \sum_{n=0}^{N} |(f, f_{N,n})_E|^2 \\
&\leqslant \left(\frac{1}{\lambda_{N,0}} - 1 \right) \|f\|_E^2.
\end{aligned}$$

Therefore, $\|T_N\| \leqslant \sqrt{(1/\lambda_{N,0} - 1)}$ and the equality is achieved by $f = f_{N,0}$.

From Theorem 2.1(e),

$$\lim_{N \to \infty} \|T_N\| = \infty.$$

Now, from the uniform boundedness principles (Banach–Steinhaus Theorem) [3, p. 43], we conclude that the set B is of first category in $H^2(D)$.

5. PROOF OF THEOREM 1.3

We prove by contradiction. Suppose that $\|p_n\| \leqslant M < \infty$. Then there exists a subsequence $\{p_{n_j}\}$ and $F \in H^2(D)$ such that

$$p_{n_j} \to F \text{ weakly.}$$

(This follows from the fact that a bounded ball in $H^2(D)$ is weak* compact).

We claim that $g = F$ a.e. on E.

To see this, let $v \in C^\infty(T)$ with support $S(v) \subset E$. Then its Fourier series converges uniformly to v. We write

$$v = \sum_{n=0}^{\infty} c_n e^{int} + \sum_{n=-\infty}^{-1} c_n e^{int} = \varphi + \phi. \tag{5.1}$$

Then $\varphi \in H^2(D)$ and ϕ is orthogonal to $H^2(D)$: $(\phi, f) = 0$ for every $f \in H^2(D)$. Write

$$(v, F-g)_E = (v, F-p_{n_j})_E + (v, p_{n_j} - g)_E.$$

From Schwarz inequality and (1.1), as $n_j \to \infty$,

$$|(v, p_{n_j} - g)_E| \leqslant \|v\| \cdot \|p_{n_j} - g\|_E \to 0, \tag{5.2}$$

and

$$\begin{aligned}(v, F-p_{n_j})_E &= (v, F-p_{n_j}) \qquad (\text{since } S(v) \subset E)\\ &= (\varphi+\phi, F-p_{n_j})\\ &= (\varphi, F-p_{n_j}) + (\phi, F-p_{n_j})\\ &= (\varphi, F-p_{n_j}) \qquad (\text{orthogonality of } \phi \text{ to } H^2(D))\\ &\to 0 \qquad (p_{n_j} \to F \text{ weakly}).\end{aligned} \tag{5.3}$$

Thus, from (5.1), (5.2), and (5.3), we deduce that $(v, F-g) = 0$ for all such v. This implies that $g = F$ a.e. on E and thus contradicts the assumption that $g \neq f_{|E}$ for any f in $H^2(D)$.

6. THE SPECTRUM IN THE LIMITING CASE: $N = \infty$

We will establish

THEOREM 6.1. *Let $m(E) < 2\pi$. Then the operator $P(E)$ has only continuous spectrum.*

We recall, from (2.2), that

$$P(E)\, f = \sum_{n=0}^{\infty} \hat{f}_E(n)\, z^n$$
$$= \frac{1}{2\pi} \int_E f(w)/(1-\bar{w}z)\, |dw|.$$

We begin with some elementary properties of $P(E)$.

LEMMA 6.2. *Let $f \in L^2(E)$. Then*

(a) *$P(E)$ is a bounded, Hermitian and non-negative operator from $L^2(E)$ to $H^2(E)$.*

(b) *if $f \in H^2(D)$ and c is a constant, then*

$$P(E)\, f = cf + \frac{1}{2\pi} \int_T (\chi(w)-c)\, f(w)/(1-\bar{w}z)\, |dw|,$$

where χ is the characteristic function of the arc E.

The proof of (a) follows from (2.4) by setting $N = \infty$.

The identity in (b) is an immediate consequence of the fact that $1/(1-w\bar{z})$ is the reproducing kernel for $H^2(D)$:

$$f(z) = (1/2\pi) \int_T f(w)/(1-z\bar{w})\, |dw|.$$

LEMMA 6.3. *Let $f \in H^2(D)$ and $g \in L^\infty(T)$ be real. If $g \neq 0$ a.e. and*

$$\int_T g(w)\, f(w)/(1-\bar{w}z)\, |dw| = 0 \tag{6.1}$$

for all $|z| < 1$, then $f = 0$ a.e.

Proof. Expanding the integral in (6.1) into a power series of z, we obtain, for z in D,

$$\sum_{n=0}^{\infty} \left(\int_T g(w)\, f(w)\, \bar{w}^n\, |dw| \right) z^n = \int_T g(w)\, f(w)/(1-\bar{w}z)\, |dw| = 0.$$

This implies

$$\int_0^{2\pi} g(e^{it})\, f(e^{it})\, e^{-int}\, dt = 0$$

for all $n \geqslant 0$; hence, the Fourier series of gf is of the form

$$g(e^{it})\, f(e^{it}) = \sum_{n=-\infty}^{-1} a_n e^{int}.$$

Let $G = \overline{fg}$. Then $G(0) = 0$ and $G \in H^2(D)$, since it clearly belongs to $L^2(T)$ and all its Fourier coefficients corresponding to the negative integers are zero. Now let $H(z) = G(z)\, f(z)$. Then $H \in H^1(D)$, $H(0) = 0$; and since $H = g\, |f|^2$, $H(z)$ is real valued for all $|z| = 1$. Thus, from the Poisson's integral formula,

$$H(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)}\, H(e^{it})\, dt \qquad (z = re^{i\theta})$$

is real valued for all z in D which clearly contradicts the analyticity of H unless H is identically zero and which, in turn, yields the desired conclusion: $f = 0$ *a.e.*

To prove Theorem 6.1, we suppose that there exists an f in $H^2(D)$ such that $P(E)\, f = \lambda f$ for some $0 < \lambda < 1$. Choosing $c = \lambda$ in Lemma 6.2(b) and applying Lemma 6.3 with $g(w) = \chi(w) - c$, we conclude that $f = 0$ *a.e.* This proves that P has no eigenvalues.

7. COMMENTS

This work is motivated in large part by problems involving the signal processings in which, for an obvious practical reason, one is required to recover the signals by approximation using the band-limited or time-limited functions (see [1, Chap. 3]) and since the norm of a signal function measures the total power of the signal, it is desirable to achieve a good approximation with small norms. In our setting, the information is provided only on the arc E (an analogue of being time-limited) and we seek to recover the information by approximation using polynomials of fixed degrees (band-limited). The fact that the norms of the approximating polynomials become unbounded, as in the cases of Theorems 1.2 and 1.3, leads naturally to the interesting question of finding an optimal approximation procedure in a situation in which only partial information is provided.

REFERENCES

1. H. Dym and H. P. McKean, "Fourier Series and Integrals," Academic Press, New York, 1972.
2. Y. Katznelson, "An Introduction to Harmonis Analysis," Dover, New York, 1976.
3. W. Rudin, "Functional Analysis," McGraw–Hill, New York, 1973.
4. W. Rudin, "Real and Complex Analysis," McGraw–Hill, New York, 1966.
5. G. Szegő, "Orthogonal Polynomials," Amer. Math. Soc., New York, 1959.
6. E. C. Titchmarsh, "The Theory of Functions," Oxford Univ. Press, London, 1932.